Asymptotic homomorphisms into the Calkin algebra *

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Abstract

Let A be a separable C^*-algebra and let B be a stable C^*-algebra with a strictly positive element. We consider the (semi)group $\mathrm{Ext}^{as}(A,B)$ (resp. $\mathrm{Ext}(A,B)$) of homotopy classes of asymptotic (resp. of genuine) homomorphisms from A to the corona algebra $M(B)/B$ and the natural map $i:\mathrm{Ext}(A,B)\longrightarrow\mathrm{Ext}^{as}(A,B)$. We show that if A is a suspension then $\mathrm{Ext}^{as}(A,B)$ coincides with E-theory of Connes and Higson and the map i is surjective. In particular any asymptotic homomorphism from SA to $M(B)/B$ is homotopic to some genuine homomorphism.

1 Introduction

Let A, B be C^*-algebras. Remind [4] that a collection of maps

$$\varphi=(\varphi_t)_{t\in[1,\infty)}:A\longrightarrow B$$

is called an asymptotic homomorphism if for every $a\in A$ the map $t\mapsto\varphi_t(a)$ is continuous and if for any $a,b\in A$, $\lambda\in\mathbf{C}$ one has

$$\lim_{t\to\infty}\|\varphi_t(ab)-\varphi_t(a)\varphi_t(b)\|=0;$$

$$\lim_{t\to\infty}\|\varphi_t(a+\lambda b)-\varphi_t(a)-\lambda\varphi_t(b)\|=0;$$

$$\lim_{t\to\infty}\|\varphi_t(a^*)-\varphi_t(a)^*\|=0.$$

Two asymptotic homomorphisms $\varphi^{(0)}$ and $\varphi^{(1)}$ are homotopic if there exists an asymptotic homomorphism Φ from A to $B\otimes C[0,1]$ such that its compositions with the evaluation maps at 0 and at 1 coincide with $\varphi^{(0)}$ and $\varphi^{(1)}$ respectively. The set of homotopy classes of asymptotic homomorphisms from A to B is denoted by $[[A,B]]$ [4, 5].

Throughout this paper we always assume that A is *separable* and that B has a strictly positive element and is stable, $B\cong B\otimes\mathcal{K}$, where $\mathcal{K}$ denotes the C^*-algebra of compact operators. We will sometimes write $B=B_1\otimes\mathcal{K}$, where $B_1=B$, to distinguish B from $B\otimes\mathcal{K}$ when necessary.

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By $\operatorname{Ext}(A,B)$ we denote the set of homotopy classes of extensions of A by B. We identify extensions with homomorphisms into the Calkin algebra $Q(B)=M(B)/B$ by the Busby invariant [3]. Two extensions $f_0, f_1 : A \longrightarrow Q(B)$ are homotopic if there exists an extension $F : A \longrightarrow Q(B\otimes C[0,1])$ such that its composition with the evaluation maps at 0 and at 1 coincide with f_0 and f_1 respectively.

Similarly we denote by $\operatorname{Ext}^{as}(A,B)$ the set of homotopy classes of asymptotic homomorphisms from A to $Q(B)$. Two asymptotic homomorphisms

$$\varphi^{(i)} = (\varphi_t^{(i)})_{t\in[1,\infty)} : A \longrightarrow Q(B), \quad i=0,1,$$

are homotopic if there exists an asymptotic homomorphism $\Phi = (\Phi_t)_{t\in[1,\infty)} : A \longrightarrow Q(B\otimes C[0,1])$ such that its compositions with the evaluation maps at 0 and at 1 coincide with $\varphi^{(0)}$ and $\varphi^{(1)}$ respectively. Asymptotic homomorphisms into $Q(B)$ are sometimes called *asymptotic extensions*.

All these sets are equipped with a natural group structure when A is a suspension, i.e. $A = SD = C_0(\mathbf{R})\otimes D$ for some C^*-algebra D.

As every genuine homomorphism can be viewed as an asymptotic one, so we have a natural map

$$i : \operatorname{Ext}(A,B) \longrightarrow \operatorname{Ext}^{as}(A,B). \tag{1}$$

It is well-known that usually there is much more asymptotic homomorphisms than genuine ones, e.g. for $A=C_0(\mathbf{R}^2)$ all genuine homomorphisms of A into $\mathcal{K}$ are homotopy trivial though the group $[[C_0(\mathbf{R}^2),\mathcal{K}]]$ coincides with $K_0(C_0(\mathbf{R}^2))=\mathbf{Z}$ via the Bott isomorphism.

The main purpose of the paper is to prove epimorphity of the map (1) when A is a suspension. This makes a contrast with the case of mappings into the compacts. As a by-product we get another description of the E-theory in terms of asymptotic extensions.

The main tool in this paper is the Connes–Higson map [4]

$$CH : \operatorname{Ext}(A,B) \longrightarrow [[SA,B]],$$

which plays an important role in E-theory. Remind that for $f\in \operatorname{Ext}(A,B)$ this map is defined by $CH(f) = (\varphi_t)_{t\in[1,\infty)}$, where φ is given by

$$\varphi_t : \alpha\otimes a \longmapsto \alpha(u_t)f'(a), \qquad a\in A,\ \alpha\in C_0(0,1).$$

Here $f' : A \longrightarrow M(B)$ is a set-theoretic lifting for $f : A\longrightarrow Q(B)$ and $u_t\in B$ is a quasicentral approximate unit [1] for $f'(A)$. We are going to show that by fine tuning of this quasicentral approximate unit one can define also a map

$$\widetilde{CH} : \operatorname{Ext}^{as}(A,B) \longrightarrow [[SA,B]]$$

extending CH and completing the commutative triangle diagram

$$\begin{tikzcd} \operatorname{Ext}(A,B) \arrow[r, "i"] \arrow[dr, "CH"'] & \operatorname{Ext}^{as}(A,B) \arrow[d, "\widetilde{CH}"] \\ & {[[SA,B]]}. \end{tikzcd}$$

We will show that the map $\widetilde{CH}$ is a isomorphism when A is a suspension.

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2 An extension of the Connes–Higson map

A useful tool for working with asymptotic homomorphisms is the possibility of discretization suggested in [12, 9, 11]. Let $\mathrm{Ext}^{as}_{discr}(A,B)$ denote the set of homotopy classes of discrete asymptotic homomorphisms $\varphi = (\varphi_n)_{n\in\mathbf{N}} : A \longrightarrow Q(B)$ with the additional crucial property suggested by Mishchenko: for every $a \in A$ one has

$$\lim_{n\to\infty} \|\varphi_{n+1}(a) - \varphi_n(a)\| = 0. \tag{2}$$

In a similar way we define a set $[[A,B]]_{discr}$ as a set of homotopy classes of discrete asymptotic homomorphisms with the property (2).

Lemma 2.1 *One has* $[[A,B]] = [[A,B]]_{discr}$*,* $\mathrm{Ext}^{as}(A,B) = \mathrm{Ext}^{as}_{discr}(A,B)$*.*

Proof. The first equality is proved in [10]. The second one can be proved in the same way. For an asymptotic homomorphism $\varphi = (\varphi_t)_{t\in[1,\infty)} : A \longrightarrow Q(B)$ one can find an infinite sequence of points $\{t_i\}_{i\in\mathbf{N}} \subset [1,\infty)$ satisfying the following properties

i) the sequence $\{t_i\}_{i\in\mathbf{N}}$ is non-decreasing and approaches infinity;

ii) for every $a \in A$ one has $\lim\limits_{i\to\infty} \sup\limits_{t\in[t_i,t_{i+1}]} \|\varphi_t(a) - \varphi_{t_i}(a)\| = 0.$

Then $\phi = (\varphi_{t_i})_{i\in\mathbf{N}}$ is a discrete asymptotic homomorphism. It is easy to see that two homotopic asymptotic homomorphisms define homotopic asymptotic homomorphisms and that two discretizations $\{t_i\}_{i\in\mathbf{N}}$ and $\{t'_i\}_{i\in\mathbf{N}}$ satisfying the above properties define homotopic discrete asymptotic homomorphisms too, hence the map $\mathrm{Ext}^{as}(A,B) \longrightarrow \mathrm{Ext}^{as}_{discr}(A,B)$ is well defined. The inverse map is given by linear interpolation of discrete asymptotic homomorphisms. □

Let $(\varphi_n)_{n\in\mathbf{N}}$ be a discrete asymptotic homomorphism and let $(m_n)_{n\in\mathbf{N}}$ be a sequence of numbers $m_n \in \mathbf{N}$. Then we call the sequence

$$(\underbrace{\varphi_1,\ldots,\varphi_1}_{m_1 \text{ times}}, \underbrace{\varphi_2,\ldots,\varphi_2}_{m_2 \text{ times}}, \varphi_3, \ldots)$$

a *reparametrization* of the sequence $(\varphi_n)_{n\in\mathbf{N}}$. It is easy to see that any reparametrization does not change the homotopy class of an asymptotic homomorphism.

Lemma 2.2 *There exists a sequence of liftings* $\varphi'_n : A \longrightarrow M(B)$ *for* φ_n*, which is continuous uniformly in* n*.*

Proof. It is easy to see [7] that

$$\lim_{n\to\infty} \sup_{n\le k<\infty} \|\varphi_k(a) - \varphi_k(b)\| \le \|a-b\|$$

for any $a, b \in A$. By the Bartle–Graves selection theorem [2], cf. [7] there exists a continuous selection $s : Q(B) \longrightarrow M(B)$. Put $\varphi'_n(a) = s\varphi_n(a)$, $a \in A$. $\square$

Now we are going to construct the map $\widetilde{CH} : \mathrm{Ext}^{as}(A, B) \longrightarrow [[SA, B]]$. Due to Lemma 2.1 it is sufficient to define the map $\widetilde{CH}$ as a map from $\mathrm{Ext}^{as}_{discr}(A, B)$ to $[[SA, B]]_{discr}$.

For $a, b \in A$, $\lambda \in \mathbf{C}$ put

$$\begin{aligned} P_n(a,b) &= \varphi_n(a)\varphi_n(b) - \varphi_n(ab); \\ L_n(a,b,\lambda) &= \varphi_n(a) + \lambda\varphi_n(b) - \varphi_n(a+\lambda b); \\ A_n(a) &= \varphi_n(a)^* - \varphi_n(a^*) \end{aligned}$$

and define $P'_n(a,b)$, $L'_n(\lambda, a)$, $A'_n(a)$ in the same way but with the liftings φ'_n instead of φ_n.

In what follows we identify $B = B_1 \otimes \mathcal{K}$ (resp. $M(B)$) with the C^*-algebra of compact (resp. adjointable) operators on the standard Hilbert C^*-module $B_1 \otimes l_2(\mathbf{N}) = l_2(B_1)$ and use the notion of diagonal operators in B and $M(B)$ in this sense. The following Lemma shows how one has to choose a quasicentral approximate unit that makes it possible to define the map $\widetilde{CH}$.

Lemma 2.3 *Let $(\varphi_n)_{n\in\mathbf{N}} : A \longrightarrow Q(B_1 \otimes \mathcal{K})$ be a discrete asymptotic homomorphism. Then there exists a reparametrization of $(\varphi_n)_{n\in\mathbf{N}}$ and an approximate unit $(u_n)_{n\in\mathbf{N}} \subset B_1 \otimes \mathcal{K}$ with the following properties:*

i) for any $a \in A$ one has

$$\lim_{n\to\infty} \|[\varphi'_n(a), u_n]\| = 0;$$

ii) for any $\alpha \in C_0(0,1)$, for any $a, b \in A$, $\lambda \in \mathbf{C}$ one has

$$\lim_{n\to\infty} \|\alpha(u_n)P'_n(a,b)\| = \lim_{n\to\infty} \|\alpha(u_n)L'_n(a,b,\lambda)\| = \lim_{n\to\infty} \|\alpha(u_n)A'_n(a)\| = 0;$$

iii) $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$*;*

iv) every u_n is a diagonal operator, $u_n = \mathrm{diag}\{u^1_n, u^2_n, \ldots\}$, where diagonal entries u^i_n belong to B_1 and

$$\lim_{i\to\infty} \sup_n \|u^{i+1}_n - u^i_n\| = 0.$$

Proof. Let $\{F_n\}_{n\in\mathbf{N}}$ be a generating system for A [4]. This means that every $F_n \subset A$ is compact, $\ldots \subset F_n \subset F_{n+1} \subset \ldots$, $\cup_n F_n$ is dense in A and one has

$$F_n \cdot F_n \subset F_{n+m(n)}; \quad F_n + \lambda F_n \subset F_{n+m(n)}, \quad (|\lambda| \le 1); \quad F^*_n \subset F_{n+m(n)}$$

for some integer sequence $m = (m_n)_{n\in\mathbf{N}}$. Let also $\alpha_0 = e^{2\pi ix} - 1 \in C_0(0,1) \cong C_0(\mathbf{R})$ be a (multiplicative) generator for $C_0(\mathbf{R})$.

Put

$$\varepsilon_{n,k} = \sup_{a,b\in F_k, |\lambda|\leq 1} \max(\|P_n(a,b)\|, \|L_n(a,b,\lambda)\|, \|A_n(a)\|).$$

For every fixed a, b, λ the sequences $(P_n(a,b))$, $(L_n(a,b,\lambda))$ and $(A_n(a))$ vanish as n approaches infinity, but the sequence $(\varepsilon_{n,n})_{n\in\mathbf{N}}$ does not have to vanish. Nevertheless one can reparametrize the sequence $\{F_n\}$ by a sequence $k = (k_n)_{n\in\mathbf{N}}$, which approaches infinity slowly enough and such that $\varepsilon_{n,k(n)}$ vanishes as $n \to \infty$. Put $\varepsilon_n = \varepsilon_{n,k(n)}$. Then

$$\lim_{n\to\infty} \varepsilon_n = 0. \tag{3}$$

Let $e = (e_n)_{n\in\mathbf{N}} \subset B$ be an approximate unit in B and let $Conv(e)$ denote its convex hull.

By induction we can choose $u_n \in Conv(e)$ in such a way that $u_n \geq u_{n-1}$ and that the estimates

$$\|[\varphi'_n(a), u_n]\| < \varepsilon_n; \tag{4}$$

and

$$\|\alpha_0(u_n)P'_n(a,b)\| < 2\varepsilon_n; \quad \|\alpha_0(u_n)L'_n(a,b,\lambda)\| < 2\varepsilon_n; \quad \|\alpha_0(u_n)A'_n(a)\| < 2\varepsilon_n \tag{5}$$

hold for any $a, b \in F_n$ and any $|\lambda| \leq 1$.

It is easy to see that the conditions (4-5) together with Lemma 2.2 ensure the first two items of Lemma 2.3.

The above choice of $(u_n)_{n\in\mathbf{N}}$ does not yet ensure the condition $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$. To make it hold we have to renumber the sequence $(\varphi_n)_{n\in\mathbf{N}}$. At first divide every segment $[u_n, u_{n+1}]$ into n equal segments $[u_{n_i}, u_{n_{i+1}}]$, $i = 1, \ldots, n$. Then as $0 \leq u_i \leq 1$ for all i, so we get $\|u_{n_{i+1}} - u_{n_i}\| \leq \frac{1}{n}$. Finally we have to change the sequences $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ and $(u_1, u_2, u_3, \ldots)$ by the sequence $(\varphi_1, \varphi_2, \varphi_2, \varphi_3, \ldots)$, where each φ_n is repeated n times, and by the sequence $(u_{1_1}, u_{2_1}, u_{2_2}, u_{3_1}, u_{3_2}, u_{3_3}, u_{4_1}, \ldots)$ respectively.

To prove the last item of Lemma 2.3 remind that an approximate unit $(e_n)_{n\in\mathbf{N}} \in B = B_1 \otimes \mathcal{K}$ can be chosen to be diagonal, $e_n = b_n \otimes \epsilon_n$, where $(b_n)_{n\in\mathbf{N}} \subset B_1$ and $(\epsilon_n)_{n\in\mathbf{N}} \subset \mathcal{K}$ are approximate units in B_1 and in $\mathcal{K}$ respectively, so the quasicentral approximate unit $(u_n)_{n\in\mathbf{N}} \subset Conv(e)$ can be made diagonal as well, with diagonal entries from B_1.

Let T be the right shift on the standard Hilbert C^*-module $l_2(B_1) = B_1 \otimes l_2(\mathbf{N})$, $T \in M(\mathcal{K}) \subset M(B_1 \otimes \mathcal{K})$. We can join S to the sets $\varphi'_n(F_n)$ in (4) when constructing the sequence (u_n). Then the sequence $[T, u_n] \in B_1 \otimes \mathcal{K}$ would vanish as n approaches infinity. Hence

$$\lim_{n\to\infty} \sup_i \|u_n^{i+1} - u_n^i\| = 0 \tag{6}$$

and the operators

$$\mathrm{diag}\{u_n^2 - u_n^1, u_n^3 - u_n^2, u_n^4 - u_n^3, \ldots\}$$

are compact, so $\lim_{i\to\infty}\|u_n^{i+1}-u_n^i\|=0$. Take $\varepsilon>0$. By (6) there exists some N such that for all $n>N$ one has $\sup_i\|u_n^{i+1}-u_n^i\|<\varepsilon$. Now consider the finite number of compact operators $\mathrm{diag}\{u_n^2-u_n^1, u_n^3-u_n^2, u_n^4-u_n^3,\ldots\}$, $1\le n\le N$. Due to their compactness there exists some I such that for $i>I$ one has $\|u_n^{i+1}-u_n^i\|<\varepsilon$ for $1\le n\le N$. Therefore for $i>I$ we have $\|u_n^{i+1}-u_n^i\|<\varepsilon$ for every n, i.e. $\sup_n\|u_n^{i+1}-u_n^i\|<\varepsilon$. □

Put now

$$\widetilde{CH}(\varphi)_n(\alpha\otimes a)=\alpha(u_n)\varphi'_n(a),\qquad \alpha\in C_0(0,1),\ a\in A,$$

where $(u_n)_{n\in\mathbf{N}}$ satisfies the conditions of Lemma 2.3. Items $i)-iii)$ of Lemma 2.3 ensure that $(\widetilde{CH}(\varphi)_n)_{n\in\mathbf{N}}$ is a discrete asymptotic homomorphism from SA to B. If $(u_n)_{n\in\mathbf{N}}$ and $(v_n)_{n\in\mathbf{N}}$ are two quasicentral approximate unities satisfying Lemma 2.3 then the linear homotopy $(tu_n+(1-t)v_n)_{n\in\mathbf{N}}$ provides that the maps $\widetilde{CH}$ defined using these approximate unities are homotopic. Finally, if φ and ψ represent the same homotopy class in $\mathrm{Ext}^{as}_{discr}(A,B)$ then $\widetilde{CH}(\varphi)$ and $\widetilde{CH}(\psi)$ are homotopic. If all φ_n are constant, $\varphi_n=f:A\longrightarrow Q(B)$ with f being a genuine homomorphism, then obviously $CH(f)=\widetilde{CH}(\varphi)$, so we have

Lemma 2.4 *The map $\widetilde{CH}:\mathrm{Ext}^{as}(A,B)\longrightarrow[[SA,B]]$ is well defined and the diagram*

$$\begin{array}{ccc}\mathrm{Ext}(A,B) & \xrightarrow{i} & \mathrm{Ext}^{as}(A,B)\\ & {\scriptstyle CH}\searrow & \downarrow{\scriptstyle \widetilde{CH}}\\ & & [[SA,B]].\end{array}$$

is commutative. □

3 An inverse for $\widetilde{CH}$

Let $\alpha_0=e^{2\pi ix}-1$ be a generator for $C_0(0,1)$ and let T be the right shift on the Hilbert space $l_2(\mathbf{N})$. By $q:M(B)\longrightarrow Q(B)$ we denote the quotient map. Define a homomorphism

$$g:C_0(0,1)\longrightarrow Q(\mathcal{K})\quad\text{by}\quad g(\alpha_0)=q(T)-1.$$

Remind that B is stable and denote by $\iota:Q(B)\otimes\mathcal{K}\subset Q(B)$ the standard inclusion. Put

$$j=\iota\circ(g\otimes\mathrm{id}_B):SB\longrightarrow Q(\mathcal{K})\otimes B\subset Q(B).$$

The homomorphism j obviously induces a map

$$j_*:[[A,SB]]\longrightarrow\mathrm{Ext}^{as}(A,B).$$

Let $S:[[A,B]]\longrightarrow[[SA,SB]]$ denote the suspension map. Then the composition $M=j_*\circ S$ gives a map

$$M:[[A,B]]\longrightarrow\mathrm{Ext}^{as}(SA,B).$$

Let

$$\beta=(\beta_n)_{n\in\mathbf{N}}:C_0(\mathbf{R}^2)\longrightarrow\mathcal{K}\tag{7}$$

be a discrete asymptotic homomorphism representing a generator of $[[C_0(\mathbf{R}^2), \mathcal{K}]]$. For a discrete asymptotic extension $\varphi = (\varphi_n)_{n\in\mathbf{N}} : A \longrightarrow Q(B)$ consider its tensor product by β

$$\varphi \otimes \beta = (\varphi_n \otimes \beta_n)_{n\in\mathbf{N}} : S^2 A \longrightarrow Q(B) \otimes \mathcal{K}$$

and denote its composition with the standard inclusion $Q(B) \otimes \mathcal{K} \subset Q(B)$ by

$$Bott_1 = \iota \circ (\varphi \otimes \beta) : \mathrm{Ext}^{as}(A, B) \longrightarrow \mathrm{Ext}^{as}(S^2 A, B).$$

In a similar way define a map

$$Bott_2 : [[A, B]] \longrightarrow [[S^2 A, B]].$$

Theorem 3.1 *One has*

$$M \circ \widetilde{CH} = Bott_1; \quad \widetilde{CH} \circ M = Bott_2.$$

Proof. We start with $M \circ \widetilde{CH} = Bott_1$. Let H be the standard Hilbert C^*-module over B, $H = B \otimes l_2(\mathbf{N})$. Put $\mathcal{H} = \oplus_{n\in\mathbf{N}} H_n$, where every H_n is a copy of H. We identify the C^*-algebra of compact (resp. adjointable) operators on both H and $\mathcal{H}$ with B (resp. $M(B)$). Instead of writing formulas in $Q(B)$ we will write them in $M(B)$ and understand them modulo compacts.

Let $\varphi = (\varphi_n)_{n\in\mathbf{N}} : A \longrightarrow Q(B)$ represent an element $[\varphi] \in \mathrm{Ext}^{as}_{discr}(A, B)$ and let $\varphi'_n : A \longrightarrow M(B)$ be liftings for φ_n as in Lemma 2.2.

If $a_n : H_n \longrightarrow H_n$ is a sequence of operators then we write $(a_1 \oplus a_2 \oplus a_3 \oplus \ldots)$ for their direct sum acting on $\mathcal{H} = \oplus_{n\in\mathbf{N}} H_n$. In what follows we use a shortcut

$$\alpha(u_n)\varphi'_n(a) = a_n.$$

Let T be the right shift on $\mathcal{H}$, $T : H_n \longrightarrow H_{n+1}$.

Remind that α_0 is a generator for $C_0(0, 1)$ and that it is sufficient to define asymptotic homomorphisms on the elements of the form $\alpha \otimes a \otimes \alpha_0 \in S^2 A$.

The composition map $M \circ \widetilde{CH}(\varphi)_n : S^2 A \longrightarrow Q(B)$ acts by

$$M \circ \widetilde{CH}(\varphi)_n(\alpha \otimes a \otimes \alpha_0) = \Big(a_n \oplus a_n \oplus a_n \oplus \ldots\Big)(T - 1)$$

modulo compacts on $\mathcal{H}$.

Let

$$v_n = \Big(\mathbf{v}^1_n \oplus \mathbf{v}^2_n \oplus \mathbf{v}^3_n \oplus \ldots\Big) \in M(B_1 \otimes \mathcal{K})$$

and

$$\lambda_n = \Big(\lambda^1_n \oplus \lambda^2_n \oplus \lambda^3_n \oplus \ldots\Big) \in M(B_1 \otimes \mathcal{K})$$

be a direct sum of diagonal operators $\mathbf{v}^i_n = \mathrm{diag}\{v^i_n, v^i_n, v^i_n, \ldots\}$, $v^i_n \in B_1$, and a direct sum of scalar operators, $\lambda^i_n \in \mathbf{R}$, ($\mathbf{v}^i_n$ and λ^i_n act on H_i). Let the numbers λ^i_n satisfy the properties

i) $\lambda_n^1 = 0$ and $\lim_{i\to\infty} \lambda_n^i = 1$ for every n;

ii) $\lim_{n\to\infty} \sup_i |\lambda_n^{i+1} - \lambda_n^i| = 0$;

iii) $\lim_{n\to\infty} \sup_i |\lambda_{n+1}^i - \lambda_n^i| = 0$.

We assume that the elements v_n^i are selfadjoint and satisfy the following properties:

i) $\lim_{n\to\infty} \sup_i \|v_n^{i+1} - v_n^i\| = 0$;

ii) $\lim_{n\to\infty} \sup_i \|v_{n+1}^i - v_n^i\| = 0$;

iii) there exists a set of scalars $\lambda_n^i \in \mathbf{R}$, $n, i \in \mathbf{N}$, satisfying the conditions $i) - iii)$ above and such that

$$\lim_{n\to\infty} \|(v_n^i - \lambda_n^i)b\| = 0 \tag{8}$$

for every i and for every $b \in B_1$.

Let p be a projection onto the first coordinate in $H = B_1 \otimes l_2(\mathbf{N})$ and let $P = (p \oplus p \oplus p \oplus \ldots)$. Then $P\lambda_n^i = \mathrm{diag}\{\lambda_n^i, 0, 0, \ldots\}$ and the map β_n (7) can be written as

$$\beta_n(\alpha \otimes \alpha_0) = P \cdot \alpha(\lambda_n) \cdot (T - 1) \in 1_{B_1} \otimes \mathcal{K} \subset M(B_1 \otimes \mathcal{K})$$

and the map $Bott_1(\varphi) : S^2A \longrightarrow Q(B)$ can be written in the form

$$(Bott_1(\varphi))_n(\alpha \otimes a \otimes \alpha_0) = \Big(\alpha(\lambda_n^1)\varphi'_n(a) \oplus \alpha(\lambda_n^2)\varphi'_n(a) \oplus \alpha(\lambda_n^3)\varphi'_n(a) \oplus \ldots\Big)(T - 1).$$

Consider also the path of asymptotic homomorphisms $(\Phi_n(t))_{n\in\mathbf{N}}$, $t \in [0, 1]$, given by the formula

$$\Phi_n(t)(\alpha \otimes a \otimes \alpha_0) = \Big(\alpha(\mathbf{v}_n^1(t))\varphi'_n(a) \oplus \alpha(\mathbf{v}_n^2(t))\varphi'_n(a) \oplus \alpha(\mathbf{v}_n^3(t))\varphi'_n(a) \oplus \ldots\Big)(T - 1),$$

where for every i $v_n^i(t)$ is a piecewise linear path connecting $v_n^i(\frac{1}{k}) = v_{n-1+k}^i$, $k \in \mathbf{N}$, and $v_n^i(0) = \lambda_n^i$. In view of (8) it is easy to check that $\Phi_n(t)$ is a homotopy connecting the asymptotic homomorphisms $(Bott_1(\varphi))_n$ and $\Phi_n = \Phi_n(0)$.

One of the obvious choices for v_n^i is to put $(v_n^i)^{i\in\mathbf{N}} = (0, \frac{1}{n}, \frac{2}{n}, \ldots, 1, 1, \ldots)$. But for our purposes it is better to use another choice. We take $v_n^i = u_i^n$ for all n and i. Lemma 2.3 ensures that the properties $i) - iii)$ are satisfied.

Now we have to connect the asymptotic homomorphisms $(Bott_1(\varphi)_n)_{n\in\mathbf{N}}$ and $(M \circ \widetilde{CH}(\varphi)_n)_{n\in\mathbf{N}}$ by a homotopy in the class of asymptotic homomorphisms. In fact we are going to do more and to connect each of these asymptotic homomorphisms with a genuine homomorphism $f : S^2A \longrightarrow Q(B)$ defined modulo compacts by

$$f(\alpha \otimes a \otimes \alpha_0) = \Big(a_1 \oplus a_2 \oplus a_3 \oplus \ldots\Big) \cdot (T - 1),$$

$\alpha \in C_0(0, 1)$, $a \in A$. Lemma 2.3 ensures that f is indeed a homomorphism.

At first we connect f with $(M \circ \widetilde{CH}(\varphi))_n$ by a path $F_n(t)$, $t \in [0,1]$. Let $F_n(1) = (M \circ \widetilde{CH}(\varphi))_n$. Denote $\alpha(u_n)\varphi'_n(a)$ by a_n and put (modulo compacts)

$$F_n\left(\frac{1}{2}\right)(\alpha \otimes a \otimes \alpha_0) = \Big(\underbrace{a_n \oplus \ldots \oplus a_n}_{n \text{ times}} \oplus a_{n+1} \oplus a_{n+1} \oplus a_{n+1} \ldots\Big)(T-1),$$

$$F_n\left(\frac{1}{3}\right)(\alpha \otimes a \otimes \alpha_0) = \Big(\underbrace{a_n \oplus \ldots \oplus a_n}_{n \text{ times}} \oplus a_{n+1} \oplus a_{n+2} \oplus a_{n+2} \oplus \ldots\Big)(T-1),$$

etc. Finally put

$$F_n(0)(\alpha \otimes a \otimes \alpha_0) = \Big(\underbrace{a_n \oplus \ldots \oplus a_n}_{n \text{ times}} \oplus a_{n+1} \oplus a_{n+2} \oplus a_{n+3} \oplus \ldots\Big)(T-1)$$

and connect $F_n(1)$, $F_n(\frac{1}{2})$, $F_n(\frac{1}{3})$, $\ldots$ and $F_n(0)$ by a piecewise linear path $F_n(t)$, $t \in [0,1]$.

It is easy to see that for every $t > 0$ the sequence $(F_n(t))_{n\in\mathbf{N}}$ is an asymptotic homomorphism. And as the maps $F_n(0)$ and f differ by compacts, so they coincide as homomorphisms into $Q(B)$. Continuity in t is also easy to check. So the asymptotic homomorphism $(M \circ \widetilde{CH}(\varphi)_n)_{n\in\mathbf{N}}$ is homotopic to the homomorphism f.

Now we are going to construct a homotopy $F'_n(t)$, $t \in [0,1]$, which connects f with $(Bott_1(\varphi)_n)_{n\in\mathbf{N}}$.

For each $k \in \mathbf{N}$ consider the following sequence $(\mathbf{u}_n^k)_{n\in\mathbf{N}}$ of diagonal operators, each of which acts on the corresponding copy of $H = H_n$ in their direct sum $\mathcal{H}$:

$$\mathbf{u}_n^k = \mathrm{diag}\{u_n^1, u_n^2, \ldots, u_n^{k-1}, u_n^k, u_n^k, u_n^k \ldots\}.$$

Put $u_n(\frac{1}{k}) = \mathbf{u}_n^k$, $u_n(0) = u_n$ and connect them by a piecewise linear path $u_n(t)$, $t \in [0,1]$. Then we get a strictly continuous path of operators $u_n(t)$, which gives a homotopy

$$F'_n(t)(\alpha \otimes a \otimes \alpha_0) = \Big(\underbrace{a_{1,n}(t) \oplus \ldots \oplus a_{n,n}(t)}_{n \text{ times}} \oplus a_{n+1,n+1}(t) \oplus a_{n+2,n+2}(t) \oplus \ldots\Big)(T-1),$$

where $a_{i,n}(t) = \alpha(u_i(t))\varphi'_n(a)$.

As

$$\Phi_n(\alpha \otimes a \otimes \alpha_0) = \Big(a_{1,n}(1) \oplus a_{2,n}(1) \oplus a_{3,n}(1) \oplus \ldots\Big)(T-1),$$

so for every $\alpha \otimes a$ one has

$$\lim_{n\to\infty} \|F'_n(1)(\alpha \otimes a \otimes \alpha_0) - \Phi_n(\alpha \otimes a \otimes \alpha_0)\| = 0,$$

hence the asymptotic homomorphisms $F'_n(1)$ and Φ_n are equivalent. But we already know that Φ_n is homotopic to $Bott_1(\varphi)_n$. On the other hand, it is easy to see that $F'_n(0)$ coincides with f modulo compacts, so we can finally conclude that $M \circ \widetilde{CH} = Bott_1$ up to homotopy.

The second identity of Theorem 3.1 is much simpler to prove. For $\psi = (\psi_n)_{n\in\mathbf{N}} : A \longrightarrow B$ we have (modulo compacts)

$$M(\psi)_n(\alpha_0 \otimes a) = \Big(\psi_1(a) \oplus \psi_2(a) \oplus \psi_3(a) \oplus \ldots\Big)(T-1), \qquad a \in A.$$

But as every $\psi_n(a) \in B_1 \otimes \mathcal{K}$, i.e. is compact, so when choosing a quasicentral approximate unit $\{w_n\}_{n\in\mathbf{N}}$ for the map $(M(\psi))_{n\in\mathbf{N}}$ we can define it by

$$w_n = \left(\mathbf{w}_1^{(n)} \oplus \mathbf{w}_2^{(n)} \oplus \mathbf{w}_3^{(n)} \oplus \ldots\right),$$

where each $\mathbf{w}_i^{(n)}$ is a finite rank diagonal operator of the form

$$\mathbf{w}_i^{(n)} = \mathrm{diag}\{\underbrace{\lambda_i b_n, \ldots, \lambda_i b_n}_{m_n \text{ times}}, 0, 0, \ldots\}$$

for some numbers $(m_n)_{n\in\mathbf{N}}$, where $(b_n)_{n\in\mathbf{N}}$ is a quasicentral approximate unit for B_1 and the scalars λ_i are defined by

$$\lambda_i = \begin{cases} \frac{n-i+1}{n} & \text{for } i < n, \\ \lambda_i = 0 & \text{for } i \geq n. \end{cases}$$

But after such a choice of w_n the map $(\widetilde{CH} \circ M)(\psi)_n$ differs from the map $Bott_2(\psi)_n$ only by the presence of b_n, hence these maps are equivalent. $\square$

4 Case of A being a suspension

As there exists a homomorphism $C_0(\mathbf{R}) \longrightarrow C_0(\mathbf{R}^3) \otimes M_2$ that induces an isomorphism in K-theory and an asymptotic homomorphism $C_0(\mathbf{R}^3) \longrightarrow C_0(\mathbf{R}) \otimes \mathcal{K}$, which are inverse to each other, so the groups $[[SA, B]]$ and $[[S^3A, B]]$ are naturally isomorphic to each other and the same is true for the groups $\mathrm{Ext}^{as}(SA, B)$ and $\mathrm{Ext}^{as}(S^3A, B)$. Hence we obtain

Corollary 4.1 *If A is a suspension then the map $\widetilde{CH} : \mathrm{Ext}^{as}(A, B) \longrightarrow [[SA, B]]$ is an isomorphism.* $\square$

It was proved in [11] that if A is a suspension then the map

$$CH : \mathrm{Ext}(A, B) \longrightarrow [[SA, B]]$$

is surjective and the group $[[SA, B]]$ is contained in $\mathrm{Ext}(A, B)$ as a direct summand. Hence from Corollary 4.1 we immediately obtain

Corollary 4.2 *Let A be a suspension. Then*

i) the map $i : \mathrm{Ext}(A, B) \longrightarrow \mathrm{Ext}^{as}(A, B)$ is surjective, hence every asymptotic extension $\varphi = (\varphi_t)_{t\in[1,\infty)} : A \longrightarrow Q(B)$ is homotopic to a genuine extension;

ii) the group $\mathrm{Ext}^{as}(A, B)$ is contained in $\mathrm{Ext}(A, B)$ as a direct summand. $\square$

Problem 4.3 Is Corollary 4.2 true when A is not a suspension ?

For C^*-algebras A and B consider the set of all extensions $f : A \longrightarrow Q(B)$ that are homotopy trivial as asymptotic homomorphisms and denote by $\mathrm{Ext}^{ph}(A, B)$ the set of homotopy classes of such homomorphisms. As usual this set becomes a group when A is a suspension. We call the elements of $\mathrm{Ext}^{ph}(A, B)$ *phantom* extensions because they constitute the part in $\mathrm{Ext}(A, B)$ which vanishes under the suspension map $S : \mathrm{Ext}(A, B) \longrightarrow \mathrm{Ext}(SA, SB)$, cf. [6].

Corollary 4.4 *If A is a second suspension then there is a natural decomposition*

$$\mathrm{Ext}(A, B) = \mathrm{Ext}^{ph}(A, B) \oplus \mathrm{Ext}^{as}(A, B). \qquad \square$$

Remark 4.5 If A is both a nuclear C^*-algebra and a suspension then the groups $\mathrm{Ext}(A, B)$ and $[[A, B]]$ coincide [8], therefore there is a one-to-one correspondence between homotopy classes of genuine and asymptotic homomorphisms into the Calkin algebras $Q(B)$ and one has $\mathrm{Ext}^{ph}(A, B) = 0$.

Problem 4.6 Does there exist a separable C^*-algebra A such that the $\mathrm{Ext}^{ph}(A, B)$ is non-zero for some B ?

Our definition of homotopy in $\mathrm{Ext}^{as}(A, B)$ is weaker than the homotopy of asymptotic homomorphisms in $[[A, Q(B)]]$, so there is a surjective map

$$p : [[A, Q(B)]] \longrightarrow \mathrm{Ext}^{as}(A, B). \tag{9}$$

It would be interesting to compare the composition $\widetilde{CH} \circ p$ with the would-be boundary map $[[A, Q(B)]] \longrightarrow [[SA, B]]$ which would exist if the exact sequences of the E-theory could be generalized to the non-separable short exact sequence $B \longrightarrow M(B) \longrightarrow Q(B)$.

Problem 4.7 Is the map p (9) injective ?

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